

THE ALMOST PRODUCT STRUCTURE OF NEWTON STRATA IN THE DEFORMATION SPACE OF A BARSOTTI-TATE GROUP WITH CRYSTALLINE TATE TENSORS

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ABSTRACT. In this paper, we construct the almost product structure of the minimal Newton stratum in deformation spaces of Barsotti-Tate groups with crystalline Tate tensors, similar to Oort's and Mantovan's construction for Shimura varieties of PEL-type. It allows us to describe the geometry of the Newton stratum in terms of the geometry of two simpler objects, the central leaf and the isogeny leaf. This yields the dimension and the closure relations of the Newton strata in the deformation space. In particular, their nonemptiness shows that a generalisation of Grothendieck's conjecture of deformations of Barsotti-Tate groups with given Newton polygon holds.

As an application, we determine analogous geometric properties of the Newton stratification of Shimura varieties of Hodge type and prove the equidimensionality of Rapoport-Zink spaces of Hodge type.

1. INTRODUCTION

Throughout the paper, let k be an algebraically closed field of characteristic $p > 2$. We denote by W the its ring of Witt vectors, $L := W[\frac{1}{p}]$ and by σ the Frobenius on K, W or k .

Let G be a reductive group over \mathbb{Z}_p , \mathbf{b} a σ -conjugacy class in $G(\widehat{\mathbb{Q}}_p^{\text{nr}})$ and μ a cocharacter of G such that $(G, \mathbf{b}, -\mu)$ is an integral local Shimura datum of Hodge type in the sense of [Kim] Def. 2.5.10. We fix a faithful representation $i : G \hookrightarrow \text{GL}(M)$ such that μ acts with weights 0 and 1 on M and a family \underline{s} of tensors of M such that G is the stabiliser of \underline{s} . Let (X, \underline{t}) be a Barsotti-Tate group with crystalline Tate tensors over k such that there exists an isomorphism of the Dieudonné module of X with $M \otimes_{\mathbb{Z}_p} W$ identifying its Frobenius with an element of the σ -conjugacy class $i(\mathbf{b})$ and the crystalline Tate tensors with $\underline{s} \otimes 1$. We denote by $\mathfrak{Def}_G(X)$ the formal subscheme of the deformation space $\mathfrak{Def}(X)$ of X cut out by i , whose power series-valued points can be regarded as those deformations where the crystalline Tate tensors \underline{t} extend to the whole group ([Fal99], [Moo98]). The space $\mathfrak{Def}_G(X)$ describes the local geometry of the Rapoport-Zink space associated to $(G, \mathbf{b}, -\mu)$ defined by Kim in [Kim] (see also [HP] for an alternative definition), and in the case where $(G, \mathbf{b}, -\mu)$ is induced by a Shimura datum of Hodge type also the local geometry of the canonical integral model of the corresponding Shimura variety ([Kis10] § 2.3).

1.1. Results on deformation spaces. From now on we change the notation as follows. Let $\mathfrak{Def}(X)$ and $\mathfrak{Def}_G(X)$ denote the algebraisations of the special fibres of the respective formal schemes. The “universal deformation” of $(\underline{X}, \underline{t})$ algebraises to a Barsotti-Tate group with crystalline Tate tensors $(\mathcal{X}^{\text{univ}}, \underline{t}^{\text{univ}})$ over $\mathfrak{Def}_G(X)$ by Messing's algebraisation result for Barsotti-Tate groups [Mes72] Lemma II.4.16

and de Jong's calculations on Dieudonné crystals in the proof of [dJ98] Prop.- 2.4.8 (cf. [HP] Rem. 2.3.5 c).

The Newton stratification is defined as the stratification corresponding to the isogeny class of the fibre of $(\mathcal{X}^{\text{univ}}, \underline{t}^{\text{univ}})$ over the geometric points of $\mathfrak{Def}_G(X)$. By Dieudonné theory the isogeny classes correspond to a certain finite subset $B(G, \mu)$ of the set $B(G)$ of σ -conjugacy classes in $G(L)$. For $\mathbf{b} \in B(G, \mu)$, denote by $\mathfrak{Def}_G(X)^{\mathbf{b}}$ the corresponding Newton stratum.

Denote by \mathbf{b}_0 the isogeny class of (X, \underline{s}) and by $\mathfrak{N}_G(X) := \mathfrak{Def}_G(X)^{\mathbf{b}_0}$ the (unique) minimal Newton stratum. We construct a surjective inseparable finite-to-finite correspondence between $\mathfrak{N}_G(X)$ and the product of its central leaf and its isogeny leaf. Here the central leaf $\mathfrak{C}_G(X) \subset \mathfrak{N}_G(X)$ is defined as the locus where the fibre of $(\mathcal{X}^{\text{univ}}, \underline{t}^{\text{univ}})$ over the geometric points is *isomorphic* to (X, \underline{t}) and the isogeny leaf $\mathfrak{I}_G(X)$ is the maximal reduced subscheme of $\mathfrak{N}_G(X)$ such that the restriction $(\mathcal{X}^{\text{univ}}, \underline{t}^{\text{univ}})|_{\mathfrak{I}_G(X)}$ is isogenous to $(X, \underline{t})|_{\mathfrak{I}_G(X)}$. If $(G, \mathbf{b}, -\mu)$ is of PEL-type and X is minimal, then this almost product structure is nothing else than the restriction of the almost product structure of Newton strata in Shimura varieties of PEL type constructed by Mantovan in [Man05].

Following Lovering's construction in [Lov], we obtain an F -isocrystal with $G_{\mathbb{Q}_p}$ -structure over \mathfrak{Def}_G . After generalising Yang's purity result for F -isocrystals ([Yan11] Thm. 1.1) to F -isocrystals with $G_{\mathbb{Q}_p}$ -structure, the geometric properties of the Newton stratification follow by a purely combinatorial argument (see e.g. [Vie] Lemma 5.12).

Theorem 1.1. *Let $\mathbf{b} \in B(G, \mu)$ such that $\mathbf{b} \geq \mathbf{b}_0$ with respect to the partial order on $B(G)$. Then*

- (1) $\mathfrak{Def}_G(X)^{\mathbf{b}}$ is non-empty and of pure dimension $\langle \rho, \mu + \bar{\nu}(\mathbf{b}) \rangle - \frac{1}{2} \text{def}(\mathbf{b})$, where ρ denotes the halfsum of positive roots of G , $\bar{\nu}(\mathbf{b})$ denotes the Newton point of \mathbf{b} and $\text{def}(\mathbf{b})$ denotes the defect of \mathbf{b} .
- (2) $\overline{\mathfrak{Def}_G(X)^{\mathbf{b}}} = \bigcup_{\mathbf{b}' \leq \mathbf{b}} \mathfrak{Def}_G(X)^{\mathbf{b}'}$.

In the cases where X is without additional structure, or when the additional structure is given by a polarisation (i.e. $G = \text{GL}_n$ or $G = \text{GSp}_n$), this result was proven by Oort ([Oor00] Th. 3.2, 3.3). If the additional structure is of PEL-type, the above theorem was proven in an earlier work of this author ([Ham15]) invoking more complicated results on Shimura varieties.

1.2. Application to Shimura varieties and Rapoport-Zink spaces. Let (G, X) be a Shimura datum of Hodge-type, $K \subset G(\mathbb{A}_f)$ a small enough compact open subgroup which is hyperspecial at p . Denote by G the reductive model of G corresponding to K_p . The existence of a canonical integral model \mathcal{S}_G of the Shimura variety $\text{Sh}_K(G, X)$ was shown by Kisin [Kis10], with some exceptions in the case $p = 2$. His construction of \mathcal{S}_G also equips it with an Abelian scheme $\mathcal{A}_G \rightarrow \mathcal{S}_G$ and crystalline Tate tensors $\underline{t}_{G,x}$ on $\mathbb{D}(\mathcal{A}_{G,x}[p^\infty])$ for every point $x \in \mathcal{S}_G(\overline{\mathbb{F}}_p)$. While the Barsotti-Tate group with crystalline Tate tensors $(\mathcal{A}_{G,x}[p^\infty], \underline{t}_x)$ depends on some choices made during the construction of \mathcal{S}_G , it induces an isocrystal with G -structure \mathcal{G}_x over $\overline{\mathbb{F}}_p$ which is independent of them. Lovering constructed an isocrystal with G -structure over the special fiber $\mathcal{S}_{G,0}$ of \mathcal{S}_G , which specialises to \mathcal{G}_x for every $x \in \mathcal{S}_G(\overline{\mathbb{F}}_p)$. Thus we have a well-defined Newton stratification on $\mathcal{S}_{G,0}$; we denote the Newton strata by $\mathcal{S}_{G,0}^{\mathbf{b}}$.

As the isomorphism of the formal neighbourhood $\mathcal{S}_{G,0,x}^\wedge$ with $\mathfrak{Def}_G(\mathcal{A}_{G,x}[p^\infty])$ given in [Kis10] § 2.3 respects the Newton stratification, the following is a direct consequence of Theorem 1.1

Theorem 1.2. *Assume that $\mathcal{S}_G^{\mathbf{b}}$ is non-empty. Then*

- (1) *$\mathcal{S}_G^{\mathbf{b}}$ is of pure dimension $\langle \rho, \mu + \overline{\nu}(\mathbf{b}) \rangle - \frac{1}{2} \text{def}(\mathbf{b})$.*
- (2) *$\mathcal{S}_G^{\mathbf{b}} = \bigcup_{\mathbf{b}' \leq \mathbf{b}} \mathcal{S}_G^{\mathbf{b}'}$.*

Another interesting consequence of the almost product structure is the equidimensionality of Rapoport-Zink spaces, which was conjectured by Rapoport in [Rap05]. It was proven by Viehmann in the cases $G = \text{GL}_n$ in [Vie08b] and $G = \text{GSp}_n$ in [Vie08a]. Our precise statement is as follows.

Theorem 1.3. *Any Rapoport-Zink space of Hodge type coming from an unramified local Shimura datum as in [Kim] is equidimensional.*

This is proven by a similar method as in [HV12] where the function field analogue is shown.

1.3. Overview. We first recall basic definitions and notation needed for the construction of the almost product structure in section 2. Moreover we generalise various results known for Barsotti-Tate groups to Barsotti-Tate groups with crystalline Tate tensors, i. a. Yang's purity theorem. In section 3 we discuss geometric properties of the central and isogeny leaves as well as the restriction of the universal deformation to them before defining the almost product structure in section 4. Finally, we deduce the above theorems in section 5.

This article was written independently from the article [Zha] of Zhang, which was put on Arxiv approximately two weeks prior to the first version of this article. In his work, he also calculates the dimension of central leaves and Newton strata in $\mathcal{S}_G^{\mathbf{b}}$. In contrast to my work, he does not consider the local geometry of the Shimura variety, but uses the geometry of stratifications of $\mathcal{S}_G^{\mathbf{b}}$ directly.

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2. PRELIMINARIES

2.1. The Newton stratification. Let H be a reductive group over \mathbb{Q}_p . We fix a quasi-split inner form H_0 together with an $H^{\text{ad}}(\overline{L})$ -conjugacy class of isomorphisms $\psi : H_{0,\overline{L}} \xrightarrow{\sim} H_{\overline{L}}$. We fix $S_0 \subset T_0 \subset B_0 \subset H_0$ where S_0 is a maximal split torus, T_0 a maximal torus and B_0 a Borel subgroup of H_0 . Denote by $\pi_1(H)$ the fundamental group of H , that is the quotient of the absolute cocharacter lattice by the coroot lattice. This construction can be made independent of the choice of a maximal torus (cf. [RR96] § 1).

There is a canonical bijection \mathbf{b} between the isomorphism classes of isocrystals with H -structure over k and the set of σ -conjugacy classes. We briefly recall the classification of σ -conjugacy classes from [Kot85], [Kot97] as presented in [RR96] § 1.

For an element $b \in H(L)$ denote by $[b] := \{gb\sigma(b)^{-1} \mid g \in G(L)\}$ the corresponding σ -conjugacy class and by $B(G)$ the set of σ -conjugacy classes of $G(L)$; this definition is independent of k by [RR96] Thm. 1.1. Kottwitz classifies the set by the two invariants $\overline{\nu}$ and κ in [Kot97] § 4.13. To define $\overline{\nu}$, denote by D the pro-torus over \mathbb{Q}_p with character group \mathbb{Q} . Kottwitz defines the slope morphism $\nu : H(L) \rightarrow \text{Hom}(D_L, G_L)$ as the unique functorial morphism such that for $H = \text{GL}(V)$ the quasi-cocharacter $\nu(b)$ induces the slope decomposition of the isocrystal $(V, b\sigma)$. Since $\nu(gb\sigma(g)^{-1}) = \text{Int}(g) \circ \nu(b)$ the slope homomorphism induces a $G(L)$ -conjugacy class $\overline{\nu}([b])$ in $\text{Hom}(D_L, H_L)$, which is called the Newton point. It can be shown that $\overline{\nu}([b])$ is invariant under σ . Rather than working with orbits, we regard $\overline{\nu}([b]) \in X_*(S_0)_{\mathbb{Q}, \text{dom}}$ as the unique dominant element in $\psi^{-1} \circ \nu([b])$. The Kottwitz point $\kappa : B(H) \rightarrow \pi_1(H)_{\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ is defined as the (unique) transformation of functors such that for $H = \mathbb{G}_m$ it is induced by $\text{val}_p : \mathbb{G}_m(\mathbb{Q}_p) \rightarrow \mathbb{Z} \cong \pi_1(\mathbb{G}_m)$. These invariants define a partial order on $B(G)$ given by $\mathbf{b}' \leq \mathbf{b}$ iff $\overline{\nu}(\mathbf{b}') \leq \overline{\nu}(\mathbf{b})$ with respect to the dominance order and $\kappa(\mathbf{b}') = \kappa(\mathbf{b})$.

We recall the basics of the Newton stratification. All statements in this paragraph are proven in [RR96] § 3. If we are given an isocrystal \mathcal{F} with G -structure over a Noetherian k -scheme S , we denote

$$S^{\mathbf{b}} = \{s \in S \mid \mathbf{b}(\mathcal{F}_{\bar{s}}) = \mathbf{b}\},$$

where \bar{s} is an arbitrary geometric point over s . We define $S^{\leq \mathbf{b}}$ and $S^{< \mathbf{b}}$ analogously. The decomposition of S as a union of $S^{\mathbf{b}}$ is called the Newton stratification. This is in general not a stratification in the strict sense, i.e. the closure of $S^{\mathbf{b}}$ does not need to be a union of strata. However, one has that $S^{\leq \mathbf{b}}$ is closed; in particular the $S^{\mathbf{b}}$ are locally closed and can be regarded as reduced subschemes.

An important geometric property of the Newton stratification is its purity. We distinguish between the following notions of purity.

Definition 2.1. Let \mathcal{F} be an isocrystal with H -structure over an \mathbb{F}_p -scheme S , where H is a reductive group over \mathbb{Q}_p . We say that the Newton stratification of S satisfies

- (1) weak purity, if the embedding $S^{\mathbf{b}} \hookrightarrow S^{\leq \mathbf{b}}$ is affine for every $\mathbf{b} \in B(H)$.
- (2) strong purity, if the embedding $S^{\leq \mathbf{b}} \setminus S^{\leq \mathbf{b}'} \hookrightarrow S^{\leq \mathbf{b}}$ is affine for every $\mathbf{b} \in B(G)$ and maximal element $\mathbf{b}' \in B(H)_{< \mathbf{b}}$.
- (3) topological strong purity, if the embedding $S^{\leq \mathbf{b}} \setminus S^{\leq \mathbf{b}'} \hookrightarrow S^{\leq \mathbf{b}}$ is pure of codimension 1 for every $\mathbf{b} \in B(H)$ and maximal element $\mathbf{b}' \in B(H)_{< \mathbf{b}}$.

It was shown by Vasiu in [Vas06] that the Newton stratification associated to an F -isocrystal without additional structure satisfies weak purity. By evaluating \mathcal{F} at a family of representations of H which separates Newton points, one easily deduces that weak purity also holds for F -isocrystals with H -structure.

It is not known whether the Newton stratification associated to an F -isocrystal without additional structure satisfies strong purity, but Yang's improvement of de Jong-Oort's purity theorem says that it satisfies topological strong purity if the base scheme is Noetherian ([Yan11] Thm. 1.1).

As topological strong purity will be needed in section 5, we prove the following proposition.

Proposition 2.2. *Let S be an \mathbb{F}_p -scheme such that every F -isocrystal over S the Newton stratification satisfies strong purity (resp. topological strong purity). Then*

for any isocrystal with H -structure over S , the Newton stratification satisfies strong purity (resp. topological strong purity).

The proof will use Viehmann's description of $B(H)_{\leq \mathbf{b}'}$, for which we need the following definition.

Definition 2.3. We consider $\nu \in X_*(S_0)_{\mathbb{Q}, \text{dom}}$ and a simple positive relative root β of H_0 .

- (1) We say that ν has a break point at β if $\langle \beta, \nu \rangle > 0$.
- (2) The value of ν at β is given by $\text{pr}_\beta(\nu) := \langle \omega_{\beta^\vee}, \nu \rangle$, where ω_{β^\vee} denotes the corresponding relative fundamental weight of $H_{0, \text{ad}}$.

Example 2.4. To motivate the definition of break points, consider $H = H_0 = \text{GL}_n$ with B_0 the Borel of upper triangular matrices and $S_0 = T_0$ the diagonal torus. We have the standard identification $X_*(S_0)_{\mathbb{Q}} = \mathbb{Q}^n$ such that $\nu \in \mathbb{Q}^n$ is dominant if $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$. The simple roots α_i of (G, B, T) are given by $\langle \alpha_i, \nu \rangle = \nu_i - \nu_{i+1}$. For simplicity we denote pr_{α_i} by pr_i .

Let P_ν be the polygon associated to ν , that is the graph of the piecewise linear function f_ν over $[0, n]$ with $f(0) = 0$ and slope ν_i over $(i-1, i)$. Then the breakpoints of P_ν are precisely at the x -coordinates i such that $\nu_i \neq \nu_{i+1}$, that is $\langle \alpha_i, \nu \rangle > 0$. Note that the y -coordinate of f at i is given by

$$f_\nu(i) = \nu_1 + \dots + \nu_i = \text{pr}_i(\nu) - \frac{i}{n} \cdot f_\nu(n).$$

So $f_\nu(i)$ and $\text{pr}_i(\nu)$ differ only by a constant depending on i and $f(n)$. In particular for $\nu, \nu' \in \mathbb{Q}^n$ we have $f_{\nu'}(i) < f_\nu(i)$ if and only if $\text{pr}_i(\nu') < \text{pr}_i(\nu)$.

Lemma 2.5 ([Vie13] Lemma 5, Rem. 6). *There exists a canonical bijection between the maximal elements of $B(H)_{< \mathbf{b}}$ and the break points of $\overline{\nu}(\mathbf{b})$. If \mathbf{b}' corresponds to a break point at β , we have*

$$B(H)_{\leq \mathbf{b}'} = \{\mathbf{b}'' \in B(H) \mid \mathbf{b}'' < \mathbf{b}, \text{pr}_\beta(\overline{\nu}(\mathbf{b}'')) < \text{pr}_\beta(\overline{\nu}(\mathbf{b}))\}$$

Remark 2.6. Technically Chai's article [Cha00], whose calculations are used in above lemma only considers the case where H is quasi-split. But as Chai remarked, this assumption is only for simplicity. Indeed, we can replace H by H_{ad} in above lemma and thus assume that H is adjoint. Then by [Kot97] p. 295f. there exists a bijection $B(H) \rightarrow B(H_0)$ which is the identity on Newton points and induces a bijection on Kottwitz points.

Proof of Proposition 2.2. None of the above notions change when we replace \mathcal{F} by the induced isocrystal with H_{ad} -structure; thus we may assume that H is semisimple.

We prove the theorem by constructing a suitable representation $\rho : H \rightarrow \text{GL}_n$ and applying the postulated strong purity of the Newton stratification of $\mathcal{F}(\rho)$ to a suitable breakpoint. Thus it suffices to show that for every simple relative root β of H there exists a representation $\rho : H \rightarrow \text{GL}_n$ and an integer $1 \leq i \leq n-1$ such that the following conditions are satisfied.

- (a) For any $\nu \in X_*(S)_{\mathbb{Q}, \text{dom}}$ with a break point at β , the quasi-cocharacter $\rho(\nu)_{\text{dom}}$ has a break point at i .
- (b) For $\nu' \leq \nu$ one has $\text{pr}_\beta(\nu') < \text{pr}_\beta(\nu)$ if and only if $\text{pr}_i(\rho(\nu')_{\text{dom}}) < \text{pr}_i(\rho(\nu)_{\text{dom}})$.

Let ω_β be the fundamental weight associated to β^\vee and N be a positive integer such that $N\omega_\beta \in X^*(S_0)$. Now let ρ be any representation such that the restriction

to S_{0L} has unique highest weight $N\omega_\beta$, and let i be the multiplicity of the weight $N\omega_\beta$ in ρ . Then

$$\rho(\nu)_{\text{dom}} = (\underbrace{N \cdot \text{pr}_\beta(\nu), \dots, N \cdot \text{pr}_\beta(\nu)}_i, \langle \lambda, \nu \rangle, \dots),$$

for some weight $\lambda \neq N\omega_\beta$ of ρ .

Thus ρ and i satisfy (b) and $\rho(\nu)_{\text{dom}}$ has a break point at i iff $\langle N\omega_\beta - \lambda, \nu \rangle > 0$. As

$$\langle N\omega_\beta - \lambda, \nu \rangle > \langle N\omega_\beta - \lambda, \langle \beta, \nu \rangle \cdot \omega_\beta^\vee \rangle = \langle \beta, \nu \rangle \cdot \langle N\omega_\beta - \lambda, \omega_\beta^\vee \rangle,$$

it suffices to show that $\langle N\omega_\beta - \lambda, \omega_\beta^\vee \rangle > 0$. Assume the contrary, i.e. $\langle N\omega_\beta - \lambda, \omega_\beta^\vee \rangle = 0$, or equivalently that $N\omega_\beta - \lambda$ is orthogonal to $N\omega_\beta$ with respect to a scalar product $(\cdot | \cdot)$ on $X^*(S)_\mathbb{R}$ which is invariant under the Weyl group. In particular, we would that get $(\lambda | \lambda) > (N\omega_\beta | N\omega_\beta)$. But since the set of weights of ρ is stable under the Weyl group of G , λ must be contained in the convex hull of the Weyl group orbit of $N\omega_\beta$ ([RR96] Lemma 2.2). Hence we get $(\lambda | \lambda) \leq (N\omega_\beta | N\omega_\beta)$. Contradiction. \square

Corollary 2.7. *Let \mathcal{F} be an isocrystal with H -structure over a Noetherian \mathbb{F}_p -scheme S . Then the Newton stratification on S satisfies topological strong purity.*

2.2. Barsotti-Tate groups with crystalline Tate tensors. We recall the definition of crystalline Tate tensors as given by Kim in [Kim]. For any object M of a rigid quasi-Abelian tensor category, we denote by M^\otimes the direct sum of any finite combination of tensor products, symmetric products, alternating products and duals of M . We define a *tensor of M* to be a morphism $s : \mathbf{1} \rightarrow M^\otimes$, where $\mathbf{1}$ denotes the unit object.

For any Barsotti-Tate group \mathcal{X} over a formally smooth \mathbb{F}_p -scheme S denote by $\mathbb{D}(\mathcal{X})$ its contravariant Dieudonné crystal as in [BBM82] § 3.3. It is a locally free crystal of $\mathcal{O}_{S/\mathbb{Z}_p, \text{CRIS}}$ -module of rank equal to the height of \mathcal{X} . The pull-back $\mathbb{D}(\mathcal{X})_S$ to the Zariski site of S is naturally equipped with the Hodge filtration $\text{Fil}_{\mathbb{D}(\mathcal{X})}^1 := \omega_{\mathcal{X}} \subset \mathbb{D}(\mathcal{X})_S$, which is Zariski-locally a direct summand of rank $\dim \mathcal{X}$. Moreover the relative Frobenius of \mathcal{X} over S induces a map $F : \mathbb{D}(\mathcal{X})^{(p)} \rightarrow \mathbb{D}(\mathcal{X})$, which is also called the Frobenius. As the relative Frobenius is an isogeny, F induces an isomorphism of isocrystals $\mathbb{D}(\mathcal{X})[\frac{1}{p}]^{(p)} \xrightarrow{\sim} \mathbb{D}(\mathcal{X})[\frac{1}{p}]$. The Frobenius induces the Hodge filtration via

$$(2.1) \quad (\text{Fil}_{\mathbb{D}(\mathcal{X})}^1)^{(p)} = \ker F|_{\mathbb{D}(\mathcal{X})_S}.$$

This additional structure induces a filtration $\text{Fil}_{\mathbb{D}(\mathcal{X})^\otimes}^\bullet$ of $\mathbb{D}(\mathcal{X})_S^\otimes$ and a morphism of isocrystals $F : \mathbb{D}(\mathcal{X})[\frac{1}{p}]^{\otimes(p)} \rightarrow \mathbb{D}(\mathcal{X})[\frac{1}{p}]^{\otimes}$.

Definition 2.8. Let \mathcal{X} be a Barsotti-Tate group over a formally smooth scheme S in characteristic p . A tensor t of $\mathbb{D}(\mathcal{X})$ is called a crystalline Tate tensor if it induces a morphism of F -isocrystals $\mathbf{1} \rightarrow \mathbb{D}(\mathcal{X})[\frac{1}{p}]^{\otimes}$, i.e. it is Frobenius equivariant.

Remark 2.9. It follows from equation 2.1 and the Frobenius equivariance that any crystalline Tate tensor is contained in $\text{Fil}_{\mathbb{D}(\mathcal{X})^\otimes}^0$.

We fix a reductive group scheme G over \mathbb{Z}_p and let M be faithful representation of G of finite rank. By [Kim] Prop. 2.1.3 (see [Kis10] Prop.1.3.2 for the original statement) there exists a family of tensors $\underline{s} \subset M^\otimes$ such that G equals the stabilizer of M . We fix such a choice of (M, \underline{s}) .

Definition 2.10. Let \mathcal{X} be a Barsotti-Tate group over a formally smooth scheme S in characteristic p . An \underline{g} -structure on \mathcal{X} is a family \underline{t} of crystalline Tate tensors of $\mathbb{D}(\mathcal{X})$, such that $(\mathbb{D}(\mathcal{X}), \underline{t})$ is fppf locally isomorphic to (M, \underline{g}) . That is, for each $(U, T, \delta) \in \text{CRIS}(S/\mathbb{Z}_p)$ there is an fppf covering $(U_i, T_i, \delta_i)_i \rightarrow (U, T, \delta)$ such that $(\mathbb{D}(\mathcal{X})_{(U_i, T_i, \delta_i)}, \underline{t}) \cong (M, \underline{g}) \otimes_{\mathcal{O}_{T_i}}$.

Remark 2.11. In the definitions 4.6 in [Kim] and 2.3.3 in [HP] they demand that the family of crystalline Tate tensors satisfies the following conditions:

- (a) $\mathbf{P} := \underline{\text{Isom}}((\mathbb{D}(\mathcal{X}), \underline{t}), (M, \underline{g}) \otimes_{\mathcal{O}_{S, \text{CRIS}}})$ is a crystal of G -torsors.
- (b) Étale locally, there exists an isomorphism $\mathbb{D}(\mathcal{X})_S \cong M \otimes_{\mathcal{O}_S}$ such that $\text{Fil}_{\mathbb{D}(\mathcal{X})}^1$ is induced by a cocharacter in a fixed conjugacy class of $\text{Hom}(G_m, G)$.

We note that the first condition is equivalent to our definition above and the second condition is satisfied if for a unique conjugacy class if S is connected. Indeed, as $\text{Fil}_{\mathbb{D}(\mathcal{X})}^1$ is locally a direct summand, the second condition is clopen on S and is satisfied for an arbitrary point of S by [Kim] Lemma 2.5.7 (2).

Given any Barsotti-Tate group with \underline{g} -structure, we obtain a functor

$$\mathcal{G} : \text{Rep}_{\mathbb{Z}_p} G \rightarrow \{ \text{strongly divisible filtered } F\text{-crystals} \}.$$

by following Lovering's construction in [Lov]. In particular, after inverting p , we obtain an F -isocrystal with $G_{\mathbb{Q}_p}$ -structure in the sense of Rapoport and Richartz.

2.3. Central leaves. We briefly recall Oort's notion of central leaves and extend it to Barsotti-Tate groups with crystalline Tate tensors. We call a Barsotti-Tate group with crystalline Tate tensors $(\mathcal{X}^{\text{univ}}, \underline{t})$ over an \mathbb{F}_p -scheme S *geometrically fiberwise constant* if each pair of fibers of $(\mathcal{X}^{\text{univ}}, \underline{t})$ over points of S becomes isomorphic after base change to a sufficiently large field. Under some mild conditions, it suffices to check whether $\mathcal{X}^{\text{univ}}$ is geometrically fiberwise constant.

Lemma 2.12. *A Barsotti-Tate group with crystalline Tate tensors $(\mathcal{X}, \underline{t})$ over a connected k -scheme S is geometrically fiberwise constant if and only if \mathcal{X} is.*

Proof. Obviously \mathcal{X} is geometrically fiberwise constant if $(\mathcal{X}, \underline{t})$ is. On the other hand, assume that \mathcal{X} is geometrically fiberwise constant. After replacing S by an irreducible component, we may assume that S is integral. By [Cha] Lemma 3.3.3 there exists a surjective morphism $T \rightarrow S$, such that \mathcal{X}_T is constant, i.e. $\mathcal{X}_T \cong X_T$ for a Barsotti-Tate group X over k . Thus we may assume without loss of generality that \mathcal{X} is constant and further that $S = \text{Spec } R$ is affine and perfect. Then by [RR96] Lemma 3.5 the morphism $u : \mathbf{1} \rightarrow \mathbb{D}(\mathcal{X})$ is constant, so in particular the isomorphism type of $(\mathcal{X}, (u_\alpha))$ over any point of S is the same. \square

In [Oor04], Oort associates to a Barsotti-Tate group \mathcal{X} over an \mathbb{F}_p -scheme S a foliation of S , such that the restriction of \mathcal{X} to the strata is geometrically fiberwise constant. We generalize this definition to Barsotti-Tate groups with crystalline Tate tensors as follows.

Definition 2.13. Let $(\mathcal{X}, \underline{t})$ be a Barsotti-Tate group with crystalline Tate tensors over a k -scheme S . For a fixed Barsotti-Tate group with crystalline Tate tensors (X_0, \underline{t}_0) over in S the central leaf of $(\mathcal{X}, \underline{t})$ is defined as

$$C_{(\mathcal{X}, \underline{t})}(X_0, \underline{t}_0) := \{s \in S \mid (\mathcal{X}^{\text{univ}}, \underline{t})_{\bar{s}} \cong (X_0, \underline{t}_0) \otimes \overline{\kappa(s)}\}$$

Proposition 2.14. *Assume that S is an excellent \mathbb{F}_p -scheme. Then the central leaf is a closed subset of a Newton stratum.*

Proof. By [Oor04] Prop. 2.2 this holds for Barsotti-Tate groups without additional structure. Now the claim follows by Lemma 2.12. \square

2.4. Transfer of quasi-isogenies. The almost product structure is based on the following construction.

Construction 2.15. Let S be an \mathbb{F}_p -scheme and \mathcal{X}, \mathcal{Y} Barsotti-Tate groups over S such that we have an isomorphism $j : \mathcal{X}[p^m] \xrightarrow{\sim} \mathcal{Y}[p^m]$ for some integer m . Let $\varphi : \mathcal{Y} \rightarrow \mathcal{Y}'$ be a quasi-isogeny such that there exist integers m_1, m_2 with $m_1 + m_2 \leq m$ such that $p^{m_1}\varphi$ and $p^{m_2}\varphi^{-1}$ are isogenies. We define the quasi-isogeny $\varphi_{\mathcal{X},j}$ as the concatenation

$$\mathcal{X} \xrightarrow{p^{-m_1}} \mathcal{X} \twoheadrightarrow \mathcal{X}/j^{-1}(\ker(p^{m_1}\varphi)).$$

If it is obvious from the context which isomorphism j is used we may also write $\varphi_{\mathcal{X},j}$ as $\varphi_{\mathcal{X}}$

To generalise this construction to Barsotti-Tate groups with crystalline Tate tensors, define the truncation $t \bmod p^m$ of a crystalline Tate tensor t as concatenation $\mathbf{1} \rightarrow \mathbb{D}(\mathcal{X})^{\otimes} \rightarrow \mathbb{D}(\mathcal{X}[p^m])^{\otimes}$. Note that $\mathbb{D}(\mathcal{X}[p^m]) \cong \mathbb{D}(\mathcal{X}) \otimes \mathcal{O}_{S, \text{CRIS}}/p^m$ ([BBM82] Thm. 3.3.2 (ii)).

Proposition 2.16. *Let S be a formally smooth \mathbb{F}_p -scheme and $(\mathcal{X}, \underline{t}), (\mathcal{Y}, \underline{u})$ be Barsotti-Tate groups with \underline{s} -structure over S such that we have an isomorphism $j : \mathcal{X}[p^m] \xrightarrow{\sim} \mathcal{Y}[p^m]$ which identifies $\underline{t} \bmod p^m$ and $\underline{u} \bmod p^m$. Let $\varphi : (\mathcal{Y}, \underline{u}) \rightarrow (\mathcal{Y}', \underline{u}')$ be a quasi-isogeny of Barsotti-Tate groups with crystalline Tate tensors such that there exist integers m_1, m_2 as above. Then $\underline{t}' := \varphi_{\mathcal{X},j*}(\underline{t})$ is a \underline{s} -structure on $\mathcal{X}' := \text{im } \varphi_{\mathcal{X},j}$.*

Proof. By construction, \underline{t}' is a family of tensors on $\mathbb{D}(\mathcal{X}')[\frac{1}{p}]$. We have to show that they factor through $\mathbb{D}(\mathcal{X})$, are F -equivariant and that $(\mathbb{D}(\mathcal{X}), \underline{t})$ is locally isomorphic to (M, \underline{s}) .

We assume without loss of generality that $S = \text{Spec } R$ is affine and denote by A the universal p -adically complete PD-extension of a smooth lift of R . It suffices to check the above assertions on A -sections. We denote

$$\begin{aligned} \mathbf{M} &:= \mathbb{D}(\mathcal{X})(A) \\ \mathbf{M}' &:= \mathbb{D}(\mathcal{X}')(A) \\ \mathbf{M}_0 &:= \mathbb{D}(\mathcal{Y})(A) \\ \mathbf{M}'_0 &:= \mathbb{D}(\mathcal{Y}')(A). \end{aligned}$$

By Remark 2.11, the scheme

$$P := \underline{\text{Isom}}_A((\mathbf{M}_0, \underline{u}), (\mathbf{M}, \underline{t}))$$

is a G_A -torsor and in particular smooth. Now j induces an isomorphism $j_{\text{CRIS}} \in P(A/p^m)$ which by the infinitesimal lifting criterion lifts to a section $\tilde{j}_{\text{CRIS}} \in P(A)$. We extend \tilde{j}_{CRIS} to an isomorphism $\mathbf{M}_0[\frac{1}{p}] \xrightarrow{\sim} \mathbf{M}[\frac{1}{p}]$.

We identify $\mathbf{M}[\frac{1}{p}] = \mathbf{M}'[\frac{1}{p}]$ and $\mathbf{M}_0[\frac{1}{p}] = \mathbf{M}'_0[\frac{1}{p}]$ using the isomorphisms induced by $\varphi_{\mathcal{X}}$ and φ . As those isogenies are compatible with crystalline Tate tensors this also identifies \underline{t} with \underline{t}' and \underline{u} with \underline{u}' . In particular, \underline{t}' is F -equivariant.

By exactness of the crystal functor ([BBM82] Cor. 4.2.8), we have

$$p^{m_1}\mathbf{M} \subset \mathbf{M}' \subset p^{-m_2}\mathbf{M}$$

$$p^{m_1}\mathbf{M}_0 \subset \mathbf{M}'_0 \subset p^{-m_2}\mathbf{M}_0$$

such that \mathbf{M}' is mapped onto \mathbf{M}'_0 by $p^{m_1} \circ \tilde{j}_{\text{CRIS}} \circ p^{-m_2} = \tilde{j}_{\text{CRIS}}$. Thus \underline{t} factorises through $\mathbb{D}(\mathcal{X}')$ and j_{CRIS} defines an isomorphism $(\mathbf{M}'_0, \underline{u}') \cong (\mathbf{M}', \underline{t}')$. In particular $(\mathbf{M}', \underline{t}')$ is locally isomorphic to $(M_A, \underline{s} \otimes 1)$. \square

3. LEAVES ON DEFORMATION SPACES

In this section we mostly work with local schemes obtained by algebraising formal schemes. Let (CNLocSch_k) be the full subcategory of local schemes whose objects are local schemes which are isomorphic to the spectrum of a complete Noetherian local k -algebra whose residue field is isomorphic to k . We denote an object of (CNLocSch_k) as (\mathfrak{S}, s) where \mathfrak{S} is the scheme and s the closed point of \mathfrak{S} . By a finite covering of (\mathfrak{S}, s) we mean a surjective finite morphism $(\mathfrak{S}', s') \rightarrow (\mathfrak{S}, s)$. For any p -power q , we denote by $F_{q/k} : (\mathfrak{S}^{(q^{-1})}, s) \rightarrow (\mathfrak{S}, s)$ the relative arithmetic Frobenius.

3.1. Central and isogeny leaves. We fix an unramified local Shimura datum $(G, \mathbf{b}, -\mu)$ in the sense of [Kim] Def. 2.5.10. That is, there exists a faithful G -representation Λ and $b \in \mathbf{b} \cap G(W)\mu(p)G(W)$ such that $\mathbf{M}_0 := W \otimes \Lambda^*$ is $b\sigma$ -stable. We choose a finite family of tensors $\underline{s} \in \mathbf{M}_0^\otimes$ such that G is their stabilizer. We denote by (X_0, \underline{t}_0) the Barsotti-Tate group with crystalline Tate tensors associated to the F -crystal with tensors $(\mathbf{M}_0, b\sigma, \underline{s})$. Let (X, \underline{t}) be a Barsotti-Tate group with crystalline Tate tensors isogenous to (X_0, \underline{t}_0) ; fix a quasi-isogeny $\rho : (X_0, \underline{t}_0) \rightarrow (X, \underline{t})$.

We first define the central and isogeny leaves on the algebraisation of the mod p deformation space $(\mathfrak{Def}(X), s_X)$ of X . That is, $(\mathfrak{Def}(X), s_X)$ represents the functor

$$\begin{aligned} (\text{CNLocSch}_k) &\rightarrow (\text{Sets})^{\text{opp}} \\ (\mathfrak{S}, s) &\mapsto \{(\mathcal{X}, \alpha) \mid \mathcal{X} \text{ is a Barsotti-Tate group over } \mathfrak{S}, \alpha : X \xrightarrow{\sim} \mathcal{X}_s\} \end{aligned}$$

Here we use that every Barsotti-Tate group over the formal spectrum of a complete Noetherian k -algebra has a unique algebraisation by [Mes72] Lemma II.4.16 to extend the deformation problem from Artin k -algebras to (CNLocSch_k) . Let $(\mathcal{X}^{\text{univ}}, \alpha^{\text{univ}})$ denote the (algebraisation of the) universal deformation of X_0 . The central leaf of $(\mathfrak{Def}(X), s_X)$ is defined as $(\mathfrak{C}(X), s_X) := C_{\mathcal{X}^{\text{univ}}}(X)$. We use analogous notions for the deformation space of X_0 .

Now let $\mathcal{M}(X_0)$ be the Rapoport-Zink space associated to X_0 and $x \in \mathcal{M}(X_0)(k)$ be the point corresponding to ρ . By the rigidity of quasi-isogenies we have a natural isomorphism between the formal neighbourhood $(\mathcal{M}(X_0)_x^\wedge, x)$ of x and $(\mathfrak{Def}(X), s_X)$ (see for example [Kim] Lemma 4.3.1). The *isogeny leaf* $(\mathfrak{I}(X), s_X)$ of $(\mathfrak{Def}(X), s_X)$ is defined as the image of the embedding $((\mathcal{M}(X_0)^{\text{red}})_x^\wedge, x) \subset (\mathcal{M}(X_0)_x^\wedge, x) = (\mathfrak{Def}(X), s_X)$. In particular, we get a universal quasi-isogeny $\rho^{\text{univ}} : X_{0, \mathfrak{I}(X)} \rightarrow \mathcal{X}_{\mathfrak{I}(X)}^{\text{univ}}$. Note that while ρ^{univ} does depend on the choice of ρ and X_0 , the isogeny leaf itself does not.

By definition, the intersection $\mathfrak{C}(X) \cap \mathfrak{I}(X)$ parametrises the deformations of X over *reduced* complete Noetherian local schemes which are geometrically fibrewise constant and isogenous to X . It follows from the proof of [Oor04] Thm. 1.3 that these conditions imply that the deformation is in fact isomorphic to X . We extract the arguments from Oort's paper for the reader's convenience.

Lemma 3.1 (cf. [Oor04] § 1.11). *Let S be a quasi-compact reduced connected scheme over k and let \mathcal{Y} and \mathcal{Y}' be geometrically fibrewise constant Barsotti-Tate*

groups over S together with a quasi-isogeny $\rho : \mathcal{Y}' \rightarrow \mathcal{Y}$. Let m_1, m be integers such that $p^{m_1}\rho$ is an isogeny with $\ker(p^{m_1}\rho) \subset \mathcal{Y}'[p^m]$ and assume that $\mathcal{Y}'[p^m]$ is defined over k . Then so is $\ker(p^{m_1}\rho)$.

Proof. We assume without loss of generality that ρ is an isogeny and denote its kernel by K . By [Oor04] Lemma 2.9 the functor

$$T \mapsto \{H \subset Y_T \mid \mathrm{rk}(H/T) = \mathrm{rk}(K/S)\}.$$

is representable by a proper k -scheme Gr . Denote by $\mathcal{H} \rightarrow Gr$ the universal object. By [Oor04] Lemma 1.10 the central leaf $C_{(Y \times Gr)/\mathcal{H}}(Y')$ is finite for any Barsotti-Tate group Y' over k . As the morphism $S \rightarrow Gr$ induced by K factors through such a central leaf and S is connected, the morphism factors through a single point, i.e. K is constant. \square

In particular, any geometrically fibrewise constant Barsotti-Tate group which is isogenous to a Barsotti-Tate group which is constant (i.e. defined over k) is constant itself. Thus we obtain

Lemma 3.2. $\mathfrak{C}(X) \cap \mathfrak{J}(X) = \{s_X\}$.

We denote by $\mathfrak{Def}_G(X) \subset \mathfrak{Def}(X)$ the algebraisation of the deformation space constructed by Faltings reduced modulo p in [Fal99]. We will recall its explicit construction in the following subsection, here we will use a moduli description (see for example [Moo98] Prop. 4.9 [Kim] Thm. 3.6). It is the formally smooth local subscheme of $\mathfrak{Def}(X)$ given by the following property. A morphism $f : (\mathrm{Spec} k[x_1, \dots, x_N], (x_1, \dots, x_N)) \rightarrow (\mathfrak{Def}(X), s_X)$ factors through $\mathfrak{Def}_G(X)$ if and only if there exist (necessarily unique) crystalline Tate tensors \underline{u} in $f^*\mathcal{X}^{\mathrm{univ}}$ which lift \underline{t} . We denote by $\underline{t}^{\mathrm{univ}}$ the universal crystalline Tate tensors of $\mathcal{X}^{\mathrm{univ}}$ over $\mathfrak{Def}_G(X)$.

Remark 3.3. Technically, the above sources only give the moduli description is only for the formal spectra of $\mathrm{Spec} k[x_1, \dots, x_N]$. But as noted in [HP] Rem. 2.3.5(c), the proof of [dJ98] Prop. 2.4.8 shows that there exists a (necessarily unique) algebraisation of the crystalline Tate tensors. Note that they induce an \underline{g} -structure. Indeed, by the rigidity of crystals it suffices to check this over the closed point, where it holds true by definition.

We define the central leaf in $\mathfrak{Def}_G(X_0)$ as $\mathfrak{C}_G(X_0) := C_{(\mathcal{X}_0^{\mathrm{univ}}, \underline{t}_0^{\mathrm{univ}})}(X_0, \underline{t}_0)$. By Lemma 2.12 we have $\mathfrak{C}_G(X_0) = \mathfrak{C}(X_0) \cap \mathfrak{Def}_G(X_0)$.

To define the isogeny leaf $\mathcal{I}_G(X_0)$ in $\mathfrak{Def}_G(X)$, repeat the above construction, replacing $\mathcal{M}(X_0)$ by the Rapoport-Zink space of Hodge type constructed in [Kim] (see also [HP]). The universal quasi-isogeny ρ_G^{univ} thus obtained is in particular compatible with the crystalline Tate-tensors. As in the case of the central leaf we have $\mathfrak{I}_G(X) = \mathfrak{J}(X) \cap \mathfrak{Def}_G(X)$, which can be used as an alternative definition if one wants to avoid using Rapoport-Zink spaces of Hodge type.

3.2. Explicit coordinates for central leaves. In this section we give an explicit description of $\mathfrak{C}_G(X_0)$ in terms of Falting's construction of the versal deformation ring of X_0 . To make the computations bearable, we restrict ourselves to certain g . For example, in the case of Barsotti-Tate groups without additional structure our restriction is equivalent to the requirement for X_0 to be completely slope divisible. As a consequence, we show that the central leaf is formally smooth of dimension $2\langle \rho, \nu \rangle$. This statement can be deduced in greater generality (see Proposition 3.9).

Our construction generalises part of Chai's work, where he gave an explicit description of $\mathfrak{C}(X_0)$ as a formal subgroup of the group of successive extensions of the isoclinic subquotients of X_0 ([Cha] § 3, [Cha05] § 7). In particular, in this case the dimension formula and formal smoothness is due to Chai.

We briefly recall Falting's construction ([Fal99] § 7, see also [Moo98] § 4.8, [Kis10] § 1.6, [Kim] § 3.2). Denote by $U_0 \subset \mathrm{GL}(\mathbf{M}_0)$ the unipotent subgroup opposite to the unipotent induced by μ and let \hat{U}_0 denote the completion of U_0 along the identity section. Consider the following objects over the ring A_0 of global sections of \hat{U}_0 .

$$\begin{aligned} \mathbf{M}_0^{\mathrm{univ}} &= A_0 \otimes_W \mathbf{M}_0 \\ \mathrm{Fil}^1 \mathbf{M}_0^{\mathrm{univ}} &= A_0 \otimes_W \mathrm{Fil}^1 \mathbf{M}_0 \\ F &:= (u_0^{-1} \cdot (1 \otimes b))\sigma, \end{aligned}$$

where $u_0 \in \hat{U}_0(A_0)$ is the “universal” section. There exists a unique connection ∇ on $\mathbf{M}_0^{\mathrm{univ}}$ which is compatible with the Frobenius F and the filtration $\mathrm{Fil}^1 \mathbf{M}_0^{\mathrm{univ}}$ ([Moo98] § 4.5). Then $(\mathbf{M}_0^{\mathrm{univ}}, \mathrm{Fil}^1 \mathbf{M}_0^{\mathrm{univ}}, F, \nabla)$ is a crystalline Dieudonne module, which gives rise to a Barsotti-Tate group \mathcal{X}_0 over A_0/p . Faltings showed that this is a universal deformation, thus $\mathrm{Spec} A_0/p \cong \mathfrak{Def}(X_0)$ and $\mathcal{X}_0 \cong \mathcal{X}_0^{\mathrm{univ}}$.

Denote $U := U_0 \cap G$ and let \hat{U} be its completion along the identity. Let A_G be the ring of global sections of \hat{U} and let $(\mathbf{M}_G^{\mathrm{univ}}, \mathrm{Fil}^1 \mathbf{M}_G^{\mathrm{univ}}, F, \nabla)$ be the restriction of $(\mathbf{M}_0^{\mathrm{univ}}, \mathrm{Fil}^1 \mathbf{M}_0^{\mathrm{univ}}, F, \nabla)$ to A_G . Then $\mathfrak{Def}(X_0, \underline{t}_0^{\mathrm{univ}}) := \mathrm{Spec} A_G$ and $\underline{t}_0^{\mathrm{univ}}$ is given by $\underline{t}_0^{\mathrm{univ}} = (1 \otimes \underline{t}_0) \in \mathbf{M}_G^{\mathrm{univ}}$.

More explicitly, let $T \subset B \subset G$ be a maximal torus and a Borel subgroup of G such that $\mu \in X_*(T)$. We denote by $(X^*(T), R, X_*(T), R^\vee)$ be the absolute root datum associated to $T \subset G$. For $\alpha \in R$ let U_α be the associated root subgroup of G_W and fix an isomorphism $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$. We denote

$$R_\mu = \{\alpha \in R \mid \langle \alpha, \mu \rangle < 0\}.$$

Then we get an isomorphism $\prod_{\alpha \in R_\mu} U_\alpha \cong U$ by multiplication. Note that U is Abelian. so we do not need to fix an order of the factors. The u_α induce an isomorphism

$$A_G \cong k[[t_\alpha \mid \alpha \in R_\mu]]$$

Moreover the canonical section $u \in U(A_G)$ is given by $u = \prod_{\alpha \in R_\mu} u_\alpha(t_\alpha)$.

Notation 3.4. We assume that the following assumptions hold for b .

- (a) We require that $b = \dot{w}\sigma(\mu)(p)$ where $\dot{w} \in (\mathrm{Norm}_G T)(W)$. In particular, we get $\nu(g) \in X_*(T)_\mathbb{Q}$.
- (b) Let M be the centralizer of ν in G . We require that $\dot{w} \in M$, i.e. \dot{w} is a representative of a Weyl group element w of (M, T) .
- (c) $\nu(b)$ is anti-dominant with respect to B .

In particular, we can write $X_0 = X_0^1 \oplus \dots \oplus X_0^m$ such that X_0^i isoclinic. Let $(\mathbf{M}_0^1, \mathrm{Fil}^1 \mathbf{M}_0^1) \oplus \dots \oplus (\mathbf{M}_0^m, \mathrm{Fil}^1 \mathbf{M}_0^m)$ the induced decomposition of $(\mathbf{M}_0, \mathrm{Fil}^1 \mathbf{M}_0)$.

Let \mathfrak{C} be an irreducible component of $\mathfrak{C}_G(X_0)$. Then the restriction $\mathcal{X}_{0|\mathfrak{C}}^{\mathrm{univ}}$ is completely slope divisible as it is completely slope divisible over the generic point ([OZ02] Prop 2.3). Its isoclinic subquotients $(\mathcal{X}_{0|\mathfrak{C}}^{\mathrm{univ}})^i$ are isomorphic to X_0^i as any completely slope divisible Barsotti-Tate group over a Henselian ring is constant by the argument of the proof of [OZ02] Prop. 3.1.

As a first step, we give thus an explicit description of the locus in $\mathfrak{Def}_G(X_0)$ where $\mathcal{X}_0^{\text{univ}}$ allows a filtration with constant isoclinic subquotients. Let

$$\begin{aligned} R_\nu &= \{\alpha \in R \mid \langle \alpha, \nu \rangle < 0\} \\ R_{\mu, \nu} &= R_\mu \cap R_\nu \end{aligned}$$

and

$$\mathfrak{Def}_{G, \nu}(X_0) := \text{Spec } k[[t_\alpha \mid \alpha \in R_{\mu, \nu}]].$$

Lemma 3.5. *Let $f : (\mathfrak{S}, s) \rightarrow (\mathfrak{Def}_G(X_0), s_{X_0})$ be a morphism corresponding to a deformation $(\mathcal{Y}, \underline{u}; \alpha)$ over \mathfrak{S} . Then \mathcal{Y} allows a filtration $0 = \mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \mathcal{Y}_2 \dots \subset \mathcal{Y}_r = \mathcal{Y}$ with constant isoclinic subquotients if and only if f factorizes over $\mathfrak{Def}_{G, \nu}(X_0)$.*

Proof. It is easy to see from the explicit construction of (\mathbf{M}_0, F) that the restriction of $\mathcal{X}_0^{\text{univ}}$ to $\mathfrak{Def}_{G, \nu}(X_0, \underline{t}_0)$ allows such a filtration.

Now assume let \mathcal{Y} be as above. It suffices to check the assertion on infinitesimal neighbourhoods, so assume $(\mathfrak{S}, s) = (\text{Spec } k[t]/t^n, (t))$. We denote by \bar{f} and $\bar{\mathcal{Y}}$ the respective restrictions to $(\bar{\mathfrak{S}}, \bar{s}) := (\text{Spec } k[t]/t^{n-1}, (t))$. By an induction argument, we may assume that \bar{f} factors through $\mathfrak{Def}_{G, \nu}(X_0)$. Let $\gamma, \bar{\gamma} \in U$ correspond to f and \bar{f} , respectively. Note that f factors through $\mathfrak{Def}_{G, \nu}(X_0)$ if and only if γ stabilises the filtration $\text{Fil}_{\mathbf{M}_0^m}^1 \otimes k[t]/t^n \subset (\text{Fil}_{\mathbf{M}_0^m}^1 \oplus \text{Fil}_{\mathbf{M}_0^{m-1}}^1) \otimes k[t]/t^n \subset \dots \subset \text{Fil}_{\mathbf{M}_0}^1 \otimes k[t]/t^n$ and fixes the subquotients. By construction we have

$$\begin{aligned} \mathbb{D}(\bar{\mathcal{Y}})_{\bar{\mathfrak{S}}} &= \mathbf{M}_0 \otimes k[t]/t^{n-1} \\ \text{Fil}_{\mathbb{D}(\bar{\mathcal{Y}})}^1 &= \text{Fil}_{\mathbf{M}_0}^1 \otimes k[t]/t^{n-1} \end{aligned}$$

Now the analogous argument of [Kim] Lemma 3.2.1 shows that \mathcal{Y} corresponds via (contravariant) Grothendieck-Messing theory to the lift $\gamma\bar{\gamma}^{-1} \cdot (\text{Fil}_{\mathbf{M}_0}^1 \otimes k[t]/t^n)$. Thus the projection $\mathcal{Y} \twoheadrightarrow \mathcal{Y}/\mathcal{Y}_{m-1} = X_0^m \times \mathfrak{S}$ induces the embedding $k[t]/t^n \otimes \text{Fil}_{\mathbf{M}_0^m}^1 \subset \gamma\bar{\gamma}^{-1}(k[t]/t^n \otimes \text{Fil}_{\mathbf{M}_0}^1)$. As U is opposite to the unipotent subgroup stabilising $\text{Fil}_{\mathbf{M}_0}^\bullet$, this implies that $\gamma\bar{\gamma}^{-1}$ fixes $\text{Fil}_{\mathbf{M}_0^m}^1 \otimes k[t]/t^n$. Repeating the argument above, we obtain $\gamma\bar{\gamma}^{-1}|_{(\text{Fil}_{\mathbf{M}_0^m}^1 \oplus \text{Fil}_{\mathbf{M}_0^{m-1}}^1) \otimes k[t]/t^n} \equiv 1 \pmod{\text{Fil}_{\mathbf{M}_0^m}^1 \otimes k[t]/t^n}$. More generally, we obtain by induction that $\gamma\bar{\gamma}^{-1}$ stabilizes the filtration $\text{Fil}_{\mathbf{M}_0}^1 \otimes k[t]/t^n \subset (\text{Fil}_{\mathbf{M}_0^m}^1 \oplus \text{Fil}_{\mathbf{M}_0^{m-1}}^1) \otimes k[t]/t^n \subset \dots \subset \text{Fil}_{\mathbf{M}_0}^1 \otimes k[t]/t^n$ and acts trivially on the subquotients. As \bar{f} factors through $\mathfrak{Def}_{G, \nu}(X_0)$, $\bar{\gamma}$ stabilises this, thus so does γ . \square

Let w be the image of \dot{w} in the Weyl group of (G, T) . We denote $\delta := w\sigma$.

Proposition 3.6. $\mathfrak{C}_G(X_0) = \text{Spec } k[[t_\alpha \mid \alpha \in R_C]]$ where

$$R_C := \{\alpha \in R_{\mu, \nu} \mid \sum_{i=1}^r \langle \delta^{-i}(\alpha), \mu \rangle \leq 0 \text{ for any } r > 0\}$$

Proof. Let η be a geometric point of $\mathfrak{Def}_{G, \nu}(X_0, \underline{t}_0)$. Then $\eta \in \mathfrak{C}_G(X)$ if and only if there exists $h \in G(W(\eta))$ such that $hg\sigma(h)^{-1} = u(\eta)^{-1}g\sigma$. As any isomorphism respects the slope filtration, h must be contained in the standard parabolic associated to M . Denote by N the unipotent radical of this parabolic, i.e. $N = \prod_{\alpha \in R_\nu} U_\alpha$. Writing $h = nm$ with $n \in N(W(k(\eta)))$ and $m \in M(W(k(\eta)))$, we see that we must have $mg\sigma(m)^{-1} = g$. Thus we may assume without loss of generality that $h = n \in N(W(k(\eta)))$. Now

$$ng\sigma(n)^{-1} = u(\eta)^{-1}g\sigma \Leftrightarrow u(\eta) = \text{Ad}(g\sigma)(n) \cdot n^{-1},$$

which can be expressed as system of equations as follows. Let w be the image of \dot{w} in the Weyl group and $\delta := w\sigma$. We write

$$n = \prod_{\alpha \in R_\nu} u_\alpha(x_\alpha)$$

for any (fixed) order of factors which satisfies the following conditions. We demand that $\langle \alpha, \nu \rangle$ is decreasing from the left-most to the right-most factor and that the factors corresponding to any δ -orbit in R_ν are consecutive. We define $\tau_\alpha := [t_\alpha(\eta)]$ for $\alpha \in R_{\mu, \nu}$ and $\tau_\alpha = 0$ otherwise, i.e. it is the U_α -component of $u(\eta)$.

Writing both sides of $u(\eta) = \text{Ad}(b\sigma)(n) \cdot n^{-1}$ as products of elements of the root groups in the order discussed above, we obtain the system of equations

$$(3.1) \quad \tau_\alpha = -x_\alpha + C_\alpha + p^{\langle \delta^{-1}(\alpha), \mu \rangle} \epsilon_\alpha x_{\delta^{-1}(\alpha)}^\sigma \text{ for every } \alpha \in R_\nu$$

where ϵ_α and C_α are defined below. First note that the lemma is equivalent to the claim that the above system of equations has a solution if and only if $\tau_\alpha = 0$ for $\alpha \notin R_C$.

The constant $\epsilon_\alpha \in W^\times$ is defined by $\text{Ad}(\dot{w})(u_\alpha(x)) = \epsilon_{w\alpha} u_{w(\alpha)}(x)$ for any $x \in W$. In order to cancel root group factors on both sides of the equation $u(\eta) = \text{Ad}(b\sigma)(n) \cdot n^{-1}$, we have to commute elements from different root groups. The commutator of these element yields an additional summand for equations corresponding to higher roots. C_α is the sum of all such contributions to the U_α -factor, more precisely it is obtained from commutators $[u_\beta(x_\beta), u_\gamma(\tau_\gamma)]$ with $i\beta + j\gamma = \alpha$ for some positive integers i, j .

Assume that there exists a system of integral solutions (x_α) to the above system of equations. We show by induction on $\langle \alpha, \nu \rangle$ that the following assertions hold.

- (a) $\text{val } x_\alpha \geq \sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle$ for every $r \geq 0$
- (b) $\tau_\alpha = 0$ if $\sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle > 0$ for some $r \geq 0$ (i.e. if $\alpha \notin R_C$). As $\text{val}_p \tau_\alpha = 0$ unless $\tau_\alpha = 0$, this assertion is equivalent to $\text{val}_p \tau_\alpha \geq \sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle$.

So assume this holds true for every $\alpha' \in R_\nu$ for which $\langle \alpha', \nu \rangle < \langle \alpha, \nu \rangle$. We first show that $\text{val}_p C_\alpha \geq \sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle$ for $r \geq 0$ which will basically allow us to ignore C_α for the rest of this proof. Indeed, the valuation of a summand corresponding to $[u_\beta(x_\beta), u_\gamma(\tau_\gamma)]$ with $i\beta + j\gamma = \alpha$ has valuation

$$i \text{val}_p(x_\beta) + j \text{val}_p(\tau_\gamma) \geq i \sum_{k=1}^r \langle \delta^{-k}(\beta), \mu \rangle + j \sum_{k=1}^r \langle \delta^{-k}(\gamma), \mu \rangle = \sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle.$$

Now we show the above claims by induction on r . For $r = 0$ the claims are obviously true, so assume $r > 0$ and that the assertions are satisfied for $r' < r$. Then by (3.1)

$$\text{val}_p(\tau_\alpha + x_\alpha) \geq \min\{C_\alpha, \langle \delta^{-1}(\alpha), \mu \rangle + \text{val}_p(x_{\delta^{-1}(\alpha)})\} \geq \sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle.$$

In the case $\langle \alpha, \mu \rangle \neq -1$, we have $\tau_\alpha = 0$ and the above assertions follow. So assume $\langle \alpha, \mu \rangle = -1$. Then (3.1) implies

$$x_\alpha^\sigma = \epsilon_{\delta(\alpha)}^{-1} \cdot p^{-\langle \alpha, \mu \rangle} \cdot (\tau_{\delta(\alpha)} + x_{\delta(\alpha)} - C_{\delta(\alpha)}),$$

and thus $\text{val}_p x_\alpha \geq -\langle \alpha, \mu \rangle = 1$. Hence if $\sum_{k=1}^r \langle \delta^{-k}(\alpha), \mu \rangle > 0$, we get $\text{val}_p \tau_\alpha > 0$ and thus $\tau_\alpha = 0$.

On the other hand, assume that $\tau_\alpha = 0$ for $\alpha \notin R_C$. We get a set of integral solutions as follows. Let N be a positive integer such that $\delta^N = 1$. Then inserting

(3.1) $N - 1$ times into itself, we obtain

$$\begin{aligned} x_\alpha &= C_\alpha + \sum_{k=1}^{N-1} \epsilon_{\alpha,k} p^{\langle \delta^{-k}(\alpha) + \dots + \delta^{-1}(\alpha), \mu \rangle} (C_{\delta^{-k}(\alpha)} - \tau_{\delta^{-k}(\alpha)})^{\sigma^k} \\ &\quad + \epsilon_{\alpha,N} p^{\langle \alpha, N\nu \rangle} x_\alpha^{\sigma^N}, \end{aligned}$$

where

$$\epsilon_{\alpha,k} := \prod_{l=0}^{k-1} \epsilon_{\delta^{-l}(\alpha)}^{\sigma^l}.$$

Multiplying by $p^{-\langle \alpha, N\nu \rangle}$, we get

$$\begin{aligned} p^{-\langle \alpha, N\nu \rangle} \cdot x_\alpha &= C_\alpha + \sum_{k=1}^{N-1} \epsilon_{\alpha,k} p^{-\langle \alpha + \dots + \delta^{-k+1}(\alpha), \mu \rangle} (C_{\delta^{-k}(\alpha)} - \tau_{\delta^{-k}(\alpha)})^{\sigma^k} \\ &\quad + \epsilon_{\alpha,N} x_\alpha^{\sigma^N}. \end{aligned}$$

As all exponents are non-negative, the standard approximation argument shows that this equation has integral solutions. For every δ -orbit in R_ν fix a representative α_0 and solve the above equation for $\alpha = \alpha_0$. We obtain x_α for the other elements in the δ -orbit by (repeatedly) solving (3.1) for x_α . \square

Now the dimension formula for $\mathfrak{C}_G(X_0)$ follows from the lemma below.

Lemma 3.7. $\#R_C = -2\langle \rho, \nu \rangle$, where ρ denotes the half-sum of positive roots in G .

Proof. First, fix a root $\alpha_0 \in R_\nu$ and consider the set $R_C \cap \delta^{\mathbb{Z}} \cdot \alpha_0$. Note that $R_C \cap \delta^{\mathbb{Z}} \cdot \alpha_0 \subset R_{\mu,\nu} \cap \delta^{\mathbb{Z}} \cdot \alpha_0 = \{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0 \mid \langle \alpha, \mu \rangle = -1\}$. In determine the order of the complement, consider the bijection Φ given by

$$\{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0 \setminus R_C \mid \langle \alpha, \mu \rangle = -1\} \rightarrow \{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0 \mid \langle \alpha, \mu \rangle = 1\}$$

$$\alpha \mapsto \delta^{-r(\alpha)}(\alpha) \text{ with } r(\alpha) \text{ minimal such that } \sum_{k=1}^{r(\alpha)} \langle \delta^{-k}(\alpha), \mu \rangle > 0.$$

To prove that Φ is a bijection, we show that

$$\Psi : \alpha \mapsto \delta^{r'(\alpha)}(\alpha) \text{ with } r'(\alpha) \text{ minimal such that } \sum_{k=0}^{r'(\alpha)} \langle \delta^k(\alpha), \mu \rangle = 0$$

is its inverse. Indeed, we have $\Psi \circ \Phi(\alpha) = \alpha$, since

$$\langle \delta^{-r(\alpha)}(\alpha) + \dots + \alpha, \mu \rangle = \langle \delta^{-r(\alpha)}(\alpha) + \dots + \delta^{-1}(\alpha), \mu \rangle + \langle \alpha, \mu \rangle = 1 + (-1) = 0$$

and if there were $r' < r(\alpha)$ such that $\sum_{k=0}^{r'} \langle \delta^{-r(\alpha)+k}(\alpha), \mu \rangle = 0$, we would obtain $\sum_{k=1}^{r(\alpha)-r'} \langle \delta^{-k}(\alpha), \mu \rangle > 0$, contradicting the definition of $r(\alpha)$. A analogous argument shows $\Phi \circ \Psi = \text{id}$. Now

$$\begin{aligned} \#R_C \cap \delta^{\mathbb{Z}} \cdot \alpha_0 &= \#\{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0 \mid \langle \alpha, \mu \rangle = -1\} - \#\{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0 \mid \langle \alpha, \mu \rangle = 1\} \\ &= - \sum_{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0} \langle \alpha, \mu \rangle \\ &= - \sum_{\alpha \in \delta^{\mathbb{Z}} \cdot \alpha_0} \langle \alpha, \nu \rangle. \end{aligned}$$

By summing over all orbits, we obtain

$$\begin{aligned} \#R_C &= - \sum_{\alpha \in R_\nu} \langle \alpha, \nu \rangle \\ &= - \sum_{\alpha > 0} \langle \alpha, \nu \rangle \\ &= - \langle 2\rho, \nu \rangle \end{aligned}$$

□

As a consequence, we obtain the following proposition.

Proposition 3.8. *If the assumptions of Notation 3.4 are satisfied, $\mathfrak{C}_G(X_0)$ is formally smooth of dimension $2\langle \rho, \overline{\nu}(b) \rangle$.*

The conditions we imposed on (X_0, \underline{t}_0) are rather restrictive. However, anticipating results from later sections of this article, we can show that the above Proposition holds in greater generality.

Proposition 3.9. *Let (X_0, \underline{t}_0) be an arbitrary Barsotti-Tate group with \underline{s} -structure over k .*

- (1) *We have $\dim \mathfrak{C}_G(X_0) = 2\langle \rho, \overline{\nu}(b) \rangle$.*
- (2) *If $(G, \mathbf{b}, -\mu)$ is induced by a Shimura datum of Hodge type, $\mathfrak{C}_G(X_0)$ is formally smooth.*

Proof. As a consequence of Proposition 4.2 (2), the dimension of the central leaf only depends on the isogeny class of (X_0, \underline{t}_0) . Thus the dimension formula follows from Proposition 3.8. If $(G, b, -\mu)$ is induced by a Shimura datum of Hodge type, $\mathfrak{D}\mathfrak{e}\mathfrak{f}_G(X_0)$ is canonically isomorphic to the formal neighbourhood of a point in a Shimura variety of Hodge type ([Kis] Prop. 1.3.10). It is shown in [Lov], that this identifies $\mathfrak{C}_G(X)$ with the formal neighbourhood of the central leaf in the Shimura variety. But central leaves in Shimura varieties are smooth, as the formal neighbourhoods of its closed points are isomorphic to each other. □

3.3. Trivialisation of p^m -torsion. By [Oor04] Thm. 1.3 there exists a finite surjective map of schemes $\tilde{\mathfrak{C}}(X_0) \rightarrow \mathfrak{C}(X_0)$ such that the pullback of $\mathcal{X}_0^{\text{univ}}[p^m]$ to $\tilde{\mathfrak{C}}(X_0)$ is constant. As we will have certain compatibility conditions, we cannot use this theorem directly. Instead, we follow Oort's construction and explain why all criteria are met. Starting point is the following observation (see [Man04] Lemma 4.1, [OZ02] Prop. 1.3 for proofs).

If \mathcal{X} is a completely slope divisible Barsotti-Tate group with isoclinic subquotients \mathcal{X}^i , then there exists a canonical isomorphism

$$\mathcal{X}[p^m]^{(p^N)} \cong \mathcal{X}^1[p^m]^{(p^N)} \oplus \dots \oplus \mathcal{X}^n[p^m]^{(p^N)}$$

for $m > 0$ and N big enough (depending on m). As these isomorphism commute with base change, we obtain a family of isomorphisms

$$i_{N,m} : F_{p^N/k}^* \mathcal{X}^1[p^m] \oplus \dots \oplus F_{p^N/k}^* \mathcal{X}^n[p^m] \xrightarrow{\sim} F_{p^N/k}^* \mathcal{X}[p^m].$$

The $i_{N,m}$ satisfy the obvious commutativity relations, allowing us to define

$$i_\infty := \varprojlim_m \varprojlim_N i_{N,m} : F_{p^\infty/k}^* (\mathcal{X}^1 \oplus \dots \oplus \mathcal{X}^n) \xrightarrow{\sim} F_{p^\infty/k}^* \mathcal{X}.$$

Lemma 3.10. *Assume that the assumptions of Notation 3.4 are satisfied.*

- (1) *There exists a canonical isomorphism $\iota_{N,m} : X_0[p^m] \times \mathfrak{C}(X_0)^{(p^{-N})} \xrightarrow{\sim} F_{p^N/k}^* \mathcal{X}_0^{\text{univ}}[p^m]$ for $m > 0$ and N big enough (depending on m) such that the obvious compatibility criteria are satisfied for varying m, N and such that $\iota_{N,m,s_0} = \alpha^{\text{univ}}$.*
- (2) *$\iota_{N,m}$ identifies $\underline{t}_0 \bmod p^m$ and $\underline{t}_0^{\text{univ}} \bmod p^m$.*

Proof. Recall that $\mathcal{X}_{|\mathfrak{C}(X_0)}^{\text{univ}}$ is completely slope divisible and that α^{univ} extends to a (necessarily unique) isomorphism j sending $X_0 \times \mathfrak{C}(X_0)$ to the direct sum of the isoclinic subquotients of $\mathcal{X}_{0|\mathfrak{C}(X)}^{\text{univ}}$. Now let for N big enough $\iota_{N,m} = i_{N,m} \circ j|_{X[p^m] \times \mathfrak{C}(X)}$.

It suffices to show the second assertion after pulling back to $\mathfrak{C}_G(X_0)^{(p^{-\infty})}$. By construction, $\mathfrak{C}_G(X_0)^{(p^{-\infty})}$ is perfect and there exists an isomorphism $\iota_\infty : X_0 \times \mathfrak{C}_G(X_0)^{(p^{-\infty})} \xrightarrow{\sim} \mathcal{X}_{0|\mathfrak{C}_G(X_0)^{(p^{-\infty})}}^{\text{univ}}$. Now $\iota_\infty(\underline{t}_0 \otimes 1)$ and $\underline{t}_0^{\text{univ}} \otimes 1$ are equal over s_{X_0} by construction, thus they are equal over $\mathfrak{C}(X_0)^{(p^\infty)}$ by [RR96] Lemma 3.5. In particular, they are equal modulo p^m . \square

Similar as in the previous chapter, the assertions of Notation 3.4 are not necessary.

Lemma 3.11. *Let (X_0, \underline{t}_0) a Barsotti-Tate group with \underline{s} -structure over k .*

- (1) *There exists a canonical isomorphism $\iota_{N,m} : X_0[p^m] \times \mathfrak{C}(X_0)^{(p^{-N})} \xrightarrow{\sim} F_{p^N/k}^* \mathcal{X}_0^{\text{univ}}[p^m]$ for $m > 0$ and N big enough (depending on m) such that the obvious compatibility criteria are satisfied for varying m, N and such that $\iota_{N,m,s_0} = \alpha^{\text{univ}}$.*
- (2) *For any $f : (\mathfrak{S}, s) \rightarrow \mathfrak{C}_G(X_0)$ with (\mathfrak{S}, s) formally smooth $\iota_{N,m}$ identifies $f^* \underline{t}_0 \bmod p^m$ and $f^* \underline{t}_0^{\text{univ}} \bmod p^m$.*

Proof. By [OZ02] Prop. 3.1 we can find a quasi-isogeny $\mathcal{Y} \rightarrow \mathcal{X}_{0|\mathfrak{C}_G(X_0)}^{\text{univ}}$ with \mathcal{Y} slope divisible. Arguing as in the proof above, we obtain a family of canonical isomorphisms $\iota'_{N,m} : Y[p^m] \times \mathfrak{C}(X_0)^{(p^{-N})} \xrightarrow{\sim} F_{p^N/k}^* \mathcal{Y}[p^m]$ for a Barsotti-Tate group Y over k . If $\ker \rho \subset \mathcal{Y}[m']$, then $\iota'_{N,m}$ induces the isomorphism $\iota_{N,m-m'}$ with the wanted properties. As the second part did not use any assumptions on (X_0, \underline{t}_0) , the proof is literally the same as above. \square

4. THE ALMOST PRODUCT STRUCTURE FOR DEFORMATION SPACES

4.1. Construction of the almost product structure. Now fix m_1, m_2 big enough such that $p^{m_1} \rho^{\text{univ}}$ and $p^{m_2} \rho^{\text{univ}-1}$ are isogenies and let $m = m_1 + m_2$. We fix a finite covering $(\tilde{\mathfrak{C}}(X_0), \tilde{s}_{X_0}) \rightarrow (\mathfrak{C}(X_0), s_{X_0})$ such that there exists an isomorphism $\iota : X_0[p^m] \times \tilde{\mathfrak{C}}(X_0) \xrightarrow{\sim} \mathcal{X}_0^{\text{univ}}[p^m]_{\tilde{\mathfrak{C}}(X_0)}$ which is compatible with α^{univ} and which can be lifted to $p^{m'}$ -torsion points after pulling back to some finite covering for any $m' > m$.

Consider the product $(\mathfrak{P}, p) := (\tilde{\mathfrak{C}}(X_0), \tilde{s}_{X_0}) \times (\mathfrak{J}(X), s_X)$ with canonical projections pr_1, pr_2 . Denote by $\tilde{\mathcal{X}}_0 := \text{pr}_1^* \mathcal{X}_0^{\text{univ}}$ and by $\tilde{\mathcal{X}}$ the image of $\tilde{\rho} := (\text{pr}_2^* \rho^{\text{univ}})_{\tilde{\mathcal{X}}_0}$. Then $\tilde{\mathcal{X}}$ induces a morphism $\pi : (\mathfrak{P}, p) \rightarrow (\mathcal{M}(X_0)_x^\wedge, x) \cong (\mathfrak{Def}(X), s_X)$. It is easy to see that π factorises through $\mathfrak{N}(X)$.

Of course π depends on the choice of $\tilde{\mathfrak{C}}(X_0)$, but for two choices $\tilde{\mathfrak{C}}(X_0)$ and $\tilde{\mathfrak{C}}(X_0)'$ such that $\tilde{\mathfrak{C}}(X_0) \rightarrow \mathfrak{C}(X_0)$ factorises through $\tilde{\mathfrak{C}}(X_0)'$ and the trivialisations of the p^m -torsion points are compatible we get a commutative diagram

$$\begin{array}{ccc}
& (\tilde{\mathfrak{C}}(X_0), \tilde{s}_{X_0}) \times (\mathfrak{I}(X), s_X) & \\
& \downarrow & \searrow \pi \\
& (\tilde{\mathfrak{C}}(X_0)', \tilde{s}'_{X_0}) \times (\mathfrak{I}(X), s_X) & \\
& \swarrow \pi' & \searrow \pi' \\
(\mathfrak{C}(X_0), s_{X_0}) & & (\mathfrak{Def}(X), s_X)
\end{array}$$

We will use this fact frequently to argue that we can enlarge $\tilde{\mathfrak{C}}(X_0)$ without losing generality.

Remark 4.1. In the case where X_0 is completely slope divisible and the coverings are as in Lemma 3.10, π is the restriction of Mantovan's morphism π_N in ([Man04]) to formal neighbourhoods.

Proposition 4.2. *Let π as above.*

- (1) *For a geometric point $\eta = (\eta_1, \eta_2) \in \mathfrak{P}$ the isomorphism class of $\tilde{\mathcal{X}}_\eta^{\text{univ}}$ coincides with the isomorphism class of $X_0/(\ker(p^m \rho^{\text{univ}}))_{\eta_2}$.*
- (2) *$\pi^{-1}(\mathfrak{C}(X)) = \tilde{\mathfrak{C}}(X_0) \times \{s_X\}$ and $\pi^{-1}(\mathfrak{I}(X)) = \{\tilde{s}_{X_0}\} \times \mathfrak{I}(X)$.*
- (3) *π is finite and surjective.*
- (4) *Let $(\mathfrak{S}, s) \in (\text{CNLocSch})$ arbitrary and P_1, P_2 two (\mathfrak{S}, s) -valued points of (\mathfrak{P}, p) . We denote $(\mathcal{Y}_i, \alpha_i) := P_i^*(\tilde{\mathcal{X}}_0, \tilde{\alpha})$ and $\rho_i := P_i^* \tilde{\rho}$. One has $\pi(P_1) = \pi(P_2)$ iff $(\mathcal{Y}_1, \alpha_1) = (\mathcal{Y}_2, \alpha_2)$ and $\rho_{1, \mathcal{Y}_1} = \rho_{2, \mathcal{Y}_2}$.*

Proof. For any natural number N we may enlarge $\mathfrak{C}(X_0)$ to assume that the pull-back of $\mathcal{X}_0^{\text{univ}}[p^{m+N}]$ and $X_0[p^{m+N}]$ to $\tilde{\mathfrak{C}}(X_0)$ are isomorphic and thus the group schemes of p^N torsion points of $\tilde{\mathcal{X}}_\eta^{\text{univ}}$ and $X_0/(\ker(p^m \rho^{\text{univ}}))_{\eta_2}$ are isomorphic. By [Oor04] Cor. 1.7 this implies that the Barsotti-Tate groups are also isomorphic if N is big enough, proving (1).

Now (1) implies that $\pi(\eta) \in \mathfrak{C}(X)$ if and only if $\eta_2 \in \mathfrak{C}(X)$, which by Lemma 3.2 is equivalent to $\eta_2 = s_X$; this shows the first assertion of (2). Certainly $\{\tilde{s}_{X_0}\} \times \mathfrak{I}(X) \subset \pi^{-1}(\mathfrak{I}(X))$. To show equality, it suffices to show that $\mathfrak{T} := \pi^{-1}(\mathfrak{I}(X)) \cap \tilde{\mathfrak{C}}(X_0) \times \{s_X\}$ equals $\{p\}$, as $\pi^{-1}(\mathfrak{I}(X))$ is closed under specialization. Indeed, we have

$$\pi(\mathfrak{T}) \subset \mathfrak{I}(X) \cap \pi(\tilde{\mathfrak{C}}(X_0) \times \{s_X\}) = \mathfrak{I}(X) \cap \mathfrak{C}(X) = \{p\},$$

which implies that $\tilde{\mathcal{X}}_{0, \mathfrak{T}}^{\text{univ}} \cong X_0$. Hence $\mathfrak{T} \subset \text{pr}_1^{-1}(s_{X_0}) = \{p\}$.

As consequence of (2) we see that $\pi^{-1}(s_X) = \{p\}$, in particular the restriction $\pi|_{\pi^{-1}(s_X)}$ is finite, as (\mathfrak{P}, p) is topologically of finite type. By [GD71] Cor. 7.4.3 this implies that π is finite. To show surjectivity, fix a morphism $f : (\mathfrak{S}, s) \rightarrow (\mathfrak{N}(X), s_X)$ with (\mathfrak{S}, s) integral. We will show that after replacing (\mathfrak{S}, s) by a finite covering, this (\mathfrak{S}, s) -point is contained in the image of π . For this we need to construct a quasi-isogeny $\varphi : \mathcal{Y} \rightarrow \mathcal{X}_{\mathfrak{S}}^{\text{univ}}$ where the geometric fibres of \mathcal{Y} are isomorphic to X_0 and $\varphi_s = \rho$ after fixing an isomorphism $X_0 \cong \mathcal{Y}_s$. Now \mathcal{Y} corresponds to an (\mathfrak{S}, s) -valued point $(\mathfrak{C}(X_0), s_{X_0})$ and after replacing (\mathfrak{S}, s) by a finite covering, we may assume that there exists a lift \tilde{y} to $(\tilde{\mathfrak{C}}(X_0), s_{X_0})$. Then φ is induced by an isogeny $\varphi_0 : X \rightarrow X_{\mathfrak{S}}^{\text{univ}}$ and $\pi(\tilde{y}, \varphi_0) = f$.

Following the construction in §2 of [OZ02], we obtain a quasi-isogeny $\psi : \mathcal{X}_{\mathfrak{S}}^{\text{univ}} \rightarrow \mathcal{Y}$ where the geometric fibres of \mathcal{Y} are isomorphic to X_0 . We sketch the construction of ψ for the reader's convenience.

- Let η be the generic point of \mathfrak{S} . Then there exists a finite extension K of $k(\eta)$ such that $\mathcal{X}_\eta^{\text{univ}} \otimes K$ is isogenous to $X \otimes K$. By replacing \mathfrak{S} by its normalisation in K , we may assume $K = k(\eta)$.
- Fix an isogeny $\psi_\eta : \mathcal{X}_\eta^{\text{univ}} \rightarrow X \otimes k(\eta)$. Let $U \subset \mathfrak{S}$ the flat locus of the closure of $\ker \psi_\eta$. Let $\psi_U : \mathcal{X}_U^{\text{univ}} \rightarrow \mathcal{X}_U^{\text{univ}} / \overline{\ker \psi_\eta}$.
- Let \mathcal{M} the scheme representing quasi-isogenies from $\mathcal{X}_\mathfrak{S}^{\text{univ}}$ of degree $d := \deg \psi_\eta$. Then ψ_U defines a section $U \rightarrow \mathcal{M}_U$. Taking the closure of its image we obtain a continuation of ψ_U to a projective \mathfrak{S} -scheme \tilde{S} .
- This isogeny descends to \mathfrak{S} by the same argument as in [OZ02] Prop. 2.7.

Choose an isomorphism $\alpha_0 : \mathcal{Y}_s \cong X_0$ and let $\rho' := \alpha_0 \circ \psi_s \circ \alpha^{\text{univ}-1}$. We need to tweak ψ such that $\rho' = \rho^{-1}$. So let $j := \rho' \circ \rho$ and choose m'_1, m'_2 such that $p^{m'_1}j$ and $p^{m'_2}j^{-1}$ are isogenies and set $m' = m'_1 + m'_2$. After replacing (\mathfrak{S}, s) by a finite covering, we may assume that $\mathcal{Y}[p^{m'}]$ is constant and choose an isomorphism $\mathcal{Y}[p^{m'}] \cong X_0[p^{m'}]$ which can be extended to truncations of any level after pulling back to a finite covering. Now $\varphi := \psi \circ j_{\mathcal{Y}}$ satisfies the conditions above.

The “if” assertion in (4) is obvious. Now assume that $\pi(P_1) = \pi(P_2)$. We denote $\varphi := \rho_2^{-1} \circ \rho_1 : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$. It suffices to show that φ is an isomorphism. Fix n_1 such that $p^{n_1}\varphi$ is an isogeny and n such that $\ker(p^{n_1}\varphi) \subset \mathcal{Y}_1[p^n]$. Replacing \mathfrak{S} by an fppf-covering if necessary, we may assume that there is an isomorphism $\iota' : X_0[p^n] \times \mathfrak{S} \rightarrow \mathcal{Y}_1[p^n]$. Then $\iota'(\ker(p^{n_1}\varphi))$ is a constant subscheme of $X_0[p^n] \times \mathfrak{S}$ by Lemma 3.1. Since φ is an isomorphism over the special point, it follows that $\iota'(\ker(p^{n_1}\varphi)) = X_0[p^{n_1}] \times \mathfrak{S}$; hence φ is an isomorphism. \square

While $\mathfrak{N}(X)$ and $\mathfrak{C}(X_0) \times \mathfrak{J}(X)$ will in general not be isomorphic, we can deduce from the above proposition that they are inseparable forms of each other. Let us make this more precise.

In the case where $\tilde{\mathfrak{C}}(X_0) = \mathfrak{C}(X_0)^{(p^{-N})}$ and ι is defined as in Lemma 3.11, denote $\pi_N := \pi$. By construction the π_N induce the same morphism $\pi_\infty : \mathfrak{P}^{(p^{-\infty})} \rightarrow \mathfrak{N}(X)^{(p^{-\infty})}$. In order to show that π_∞ is an isomorphism, we use the following lemma.

Lemma 4.3. *Any universal homeomorphism of perfect k -schemes is an isomorphism.*

Proof. We may assume that the schemes are affine, i.e. it suffices to show that any weakly subintegral extension $A \subset B$ of perfect rings is trivial. Note that as the residue fields are perfect, the extension is subintegral. If we had $A \neq B$, there exists $b \in B \setminus A$ such that $b^2, b^3 \in A$ (see e.g. [RRS96] § 1). But b can be written as product of powers of $b^{2/p}$ and $b^{3/p}$ and is thus contained in A . Contradiction. \square

Corollary 4.4. π_∞ is an isomorphism

Proof. As π_∞ is integral and surjective, it suffices to show that it is a monomorphism in the category of perfect schemes over k , which is canonically equivalent to the localisation of the category of k -schemes obtained by inverting $F_{p/k}$. A morphism is a monomorphism iff its diagonal morphism is an isomorphism. As π_N is finite, its diagonal morphisms in the categories (Sch_k) and (LocNoeth_k) coincide. Thus, it suffices to prove that π_N is a monomorphism in $(\text{LocNoeth}_k)[F_{p/k}^{-1}]$.

Let P_1, P_2 be as in part (4) of above proposition. Now $(\mathcal{Y}_1, \alpha_1) = (\mathcal{Y}_2, \alpha_2)$ is equivalent to $F_{p^N/k} \circ \text{pr}_1 \circ P_1 = F_{p^N/k} \circ \text{pr}_1 \circ P_2$. Moreover the endomorphism

$\iota_{P_2} \circ \iota_{P_1}$ of $\mathcal{Y}_1[p^m]$ becomes trivial after pullback to $\mathfrak{S}^{(p^{-\infty})}$ as it can be lifted to an endomorphism of $\mathcal{Y}_1 \mathfrak{S}^{(p^{-\infty})}$. Thus $\rho_1 = \rho_2$ implies that $F_{p/k}^{N'} \circ \text{pr}_2 \circ P_1 = F_{p/k}^{N'} \circ \text{pr}_2 \circ P_2$ for N' big enough. Hence for $n = \max\{N, N'\}$ we get $F_{p/k}^{n*} P_1 = F_{p/k}^{n*} P_2$. \square

4.2. Restriction to \mathfrak{Def}_G . Now assume that ι identifies the pullbacks of $\underline{t}_0^{\text{univ}} \bmod p^m$ and $\underline{t}_0 \bmod p^m$ to $\tilde{\mathfrak{C}}_G(X_0)$. Moreover we assume that for any $m' > m$ we can pull back to some finite covering where ι can be lifted to an isomorphism of $p^{m'}$ -torsion points which is compatible with the \underline{g} -structure.

We denote $(\mathfrak{P}_G, p) := (\tilde{\mathfrak{C}}_G(X_0), s_{X_0}) \times (\mathfrak{I}_G(X), s_X) \subset (\mathfrak{P}, p)$ and let π_G be the restriction of π to (\mathfrak{P}_G, p) .

Proposition 4.5. *The image of π_G is $\mathfrak{N}_G(X)$.*

Proof. It follows from a simple topological argument that it suffices to check the claim on points of dimension 1, as they are dense.

So let $\eta \in \mathfrak{P}_G$ of dimension one and let $P \in (\mathfrak{P}, p)(k[[t]], (t))$ be the normalisation of its closure. Then $\pi_G(P)$ factors through $\mathfrak{N}_G(X)$ by Proposition 2.16, thus $\pi_G(\eta) \in \mathfrak{N}_G(X)$. On the other hand let $f : (\text{Spec } k[[t]], (t)) \rightarrow (\mathfrak{N}_G, s_X)$. It suffices to show that the preimage of f constructed in the proof of Proposition 4.2 lies in (\mathfrak{P}_G, p) . First note that any finite extension of $k[[t]]$ we take during the construction may be chosen such that it is again isomorphic to $k[[t]]$.

In the first step of the construction we require that ψ_η is compatible with crystalline Tate tensors. Then $\psi_*(\underline{t}^{\text{univ}})$ factors over $\mathbb{D}(\mathcal{Y})^\otimes$ as this holds true in the generic fiber by construction. After replacing ψ and \mathcal{Y} such that we get the correct isogeny in the special fibre this still holds true by Proposition 2.16 if we ensure that $(\mathcal{Y}[p^m], \psi_*(\underline{t}^{\text{univ}}) \bmod p^m)$ is constant. This can be shown as in Lemma 3.10.

Thus the $(\text{Spec } k[[t]], (t))$ -valued point of $(\mathfrak{C}(X_0), s_{X_0})$ which is induced by \mathcal{Y} is a point in $(\mathfrak{C}_G(X_0), s_{X_0})$. Hence its lift \tilde{y} is contained in $(\tilde{\mathfrak{C}}_G(X_0), s_{X_0})(\text{Spec } k[[t]], (t))$. Similarly, φ_0 defines a point in $(\mathfrak{I}_G(X), s_X)$ as it respects the crystalline Tate-tensors. Thus the preimage (y, φ_{X_0}) is indeed a point of (\mathfrak{P}_G, p) . \square

The following properties of π_G are a direct consequence of the analogous properties of π .

Proposition 4.6. *Let π_G as above.*

- (1) π_G is finite and surjective. In the situation of Lemma 3.11 the perfection of π_G is an isomorphism.
- (2) $\pi_G^{-1}(\mathfrak{I}_G(X)) = \{s_{X_0}\} \times \mathfrak{I}_G(X)$ and $\pi_G^{-1}(\mathfrak{C}_G(X)) = \mathfrak{C}_G(X_0) \times \{s_X\}$.

5. DIMENSION FORMULAS

5.1. The Newton stratification on deformation spaces. We keep the notation of the previous subsection and moreover denote by b_0 the isogeny class of (X, \underline{t}) . As a consequence of Proposition 4.5 we obtain the equality

$$\dim \mathfrak{N}_G(X) = \dim \mathfrak{C}_G(X) + \dim \mathfrak{I}_G(X).$$

Using the interpretation of $\mathfrak{I}_G(X)$ as formal neighbourhood of a point in the reduced subscheme associated to the Rapoport-Zink space of Hodge type, we know that the

dimension of $\mathfrak{I}_G(X)$ is less or equal than the dimension of the Rapoport-Zink space. Thus by [Zhu],

$$\dim \mathfrak{I}_G(X) \leq \langle \rho, \mu - \overline{\nu}(b_0) \rangle - \frac{1}{2} \operatorname{def}_G(b_0).$$

As we proved in section 3.2, the dimension of the central leaf equals $2\langle \rho, \overline{\nu}(b_0) \rangle$. Thus the above inequality is equivalent to

$$\dim \mathfrak{N}_G(X) \leq \langle \rho, \mu + \overline{\nu}(b_0) \rangle - \frac{1}{2} \operatorname{def}_G(b_0).$$

This inequality can be expressed in more natural terms. For two isogeny classes $b' \leq b$, denote by $\ell[b', b]$ the maximal length of a chain $b' < b_1 < \dots < b$ of isogeny classes. Using the corrected formula of [Cha00] Thm. 7.4 (see for example [Ham15] Prop. 3.11, [Vie] Thm. 3.4), the above inequality is equivalent to

$$(5.1) \quad \operatorname{codim} \mathfrak{N}_G(X) \geq \ell[b_0, b_{\max}],$$

where $b_{\max} = [\mu(p)]$.

Now the dimension formula and closure relations of Newton strata in $\mathfrak{Def}_G(X)$ follow from this inequality and the purity of the Newton stratification by a formal argument given in [Vie] Lemma 5.12.¹

Proposition 5.1. *For any $b_0 \leq b \leq b_{\max}$*

- (1) $\mathfrak{Def}_G(X)^{\leq b}$ *is of pure codimension $\ell[b, b_{\max}]$ in $\mathfrak{Def}_G(X)$.*
 - (2) $\mathfrak{Def}_G(X)^{\leq b}$ *is the closure of $\mathfrak{Def}_G(X)^b$.*
- In particular, $\mathfrak{Def}_G(X)^b$ is non-empty.*

5.2. Global results. We obtain the following global results as easy corollaries to our considerations above.

Corollary 5.2. *The underlying reduced subscheme of a Rapoport-Zink space of Hodge type $\mathcal{M}_{G,\mu}(b)$ is equidimensional.*

Proof. The formal neighbourhood of a closed point in the underlying reduced subscheme of a Rapoport-Zink space of Hodge type is isomorphic to an isogeny leaf $\mathfrak{I}_G(X)$ (cf. section 3.1). Hence it is of dimension

$$\dim \mathfrak{I}_G(X) = \dim \mathfrak{N}_G(X, \underline{t}) - \dim \mathfrak{C}_G(X) = \langle \rho, \mu - \nu \rangle - \frac{1}{2} \operatorname{def}_G(b),$$

which coincides with the dimension of underlying reduced subscheme of the Rapoport-Zink space. \square

Let $\mathcal{S}_{G,0}$ be the special fibre of the canonical model of a Shimura variety of Hodge type. By construction, the formal neighbourhood of any point $x \in \mathcal{S}_{G,0}(\overline{\mathbb{F}}_p)$ is canonically isomorphic to some deformation space $\mathfrak{Def}_G(\mathcal{A}_x[p^\infty])$ and the isomorphism preserves the Newton stratification (see [Lov]). Thus we obtain the following corollary.

Corollary 5.3. *Let $b \in B(G, \mu)$.*

- (1) $\mathcal{S}_{G,0}^{\leq b}$ *is of pure codimension $\ell[b, b_{\max}]$ in $\mathcal{S}_{G,0}$.*
- (2) $\mathcal{S}_{G,0}^{\leq b}$ *is the closure of $\mathcal{S}_{G,0}^b$.*

¹Actually, Viehmann proves it under the assumption of strong purity. But her proof only uses topological strong purity.

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